2 Factors

2.1 Factors and Prime Numbers

A *factor* divides *exactly* into a number, leaving *no* remainder. For example, 13 is a *factor* of 26 because $26 \div 13 = 2$ leaving no remainder.

A *prime number* has only *two* factors, 1 and itself; this is how a prime number is defined.

5 is a prime number because it has only two factors, 1 and 5.

8 has factors 1, 2, 4 and 8, so it is *not* prime.

1 is *not* a prime number because it has only one factor, namely 1 itself.

Example 1

(a) List the factors of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10.

(b) Which of these numbers are *prime* numbers?

Solution

(a) This table lists the factors of these numbers:

Number	*Factors*
1	1
2	1, 2
3	1, 3
4	1, 2, 4
5	1, 5
6	1, 2, 3, 6
7	1, 7
8	1, 2, 4, 8
9	1, 3, 9
10	1, 2, 5, 10

(b) The numbers 2, 3, 5 and 7 have exactly two factors, and so only they are prime numbers.

Example 2

List the prime factors of 24.

Solution

First list all the factors of 24, and they are:

1, 2, 3, 4, 6, 8, 12, 24

Now select from this list the numbers that are prime; these are 2 and 3, and so the prime factors of 24 are 2 and 3.

Example 3

Which of the following numbers are prime numbers:

18, 45, 79 and 90 ?

Solution

The factors of 18 are 1, 2, 3, 6, 9 and 18; 18 is not a prime number.

The factors of 45 are 1, 3, 5, 9, 15 and 45; 45 is not a prime number.

The factors of 79 are 1 and 79; 79 is a prime number

The factors of 90 are 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45 and 90; 90 is not a prime number.

79 is the only prime number in the list.

Divisibility Test

If a number is divisible by **2**, it will end with 0, 2, 4, 6 or 8.

If a number is divisible by **3**, the *sum* of its digits will be a multiple of 3.

If a number is divisible by **4**, the last two digits will be a multiple of 4.

If a number is divisible by **5**, it will end in 0 or 5.

If a number is divisible by **9**, the *sum* of its digits will be a multiple of 9.

If a number is divisible by **10**, it will end in 0.

Can you find tests for divisibility by other numbers?

Exercises

1. (a) List all the factors of each of the following numbers:

   11, 12, 13, 14, 15, 16, 17, 18, 19, 20

   (b) Which of these numbers are prime?

2. Explain why 99 is *not* a prime number.

3. Which of the following are prime numbers:

   33, 35, 37, 39 ?

4. Find the prime factors of 72.

5. (a) Find the prime factors of 40.

   (b) Find the prime factors of 70.

   (c) Which prime factors do 40 and 70 have in common?

6. Find the prime factors that 48 and 54 have in common.

7. A number has prime factors 2, 5 and 7. Which is the *smallest* number that has these prime factors?

8. The first 5 prime numbers are 2, 3, 5, 7 and 11. Which is the *smallest* number that has these prime factors?

9. Write down the first *two* prime numbers which are greater than 100.

10. Which is the first prime number that is greater than 200?

2.2 Prime Factors

A *factor tree* may be used to help find the prime factors of a number.

Example 1

Draw a factor tree for the number 36.

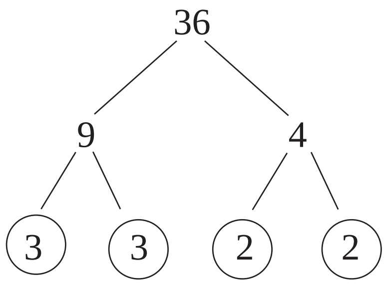
Solution

Start with 36 and then:

split 36 into numbers 9 and 4 that multiply to give 36 as shown in the factor tree opposite;

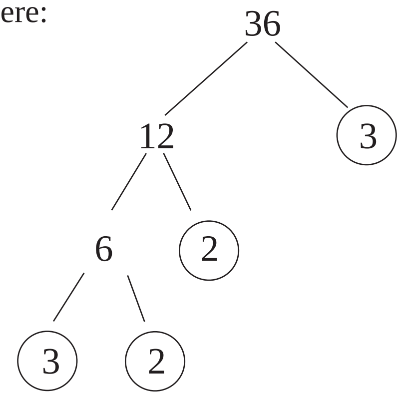
repeat for the 9 and the 4, as shown on the factor tree.

The factor tree is now complete because the numbers at the ends of the branches are prime numbers; the prime numbers have been ringed.



Another possible factor tree for 36 is shown here:

On the factor tree we only put a ring around the prime numbers.



Note that, at the end of the branches, both the numbers 2 and 3 appear twice.

The prime factors of 36 are 3, 2, 2 and 3.

In ascending order, the prime factors of 36 are 2, 2, 3, 3.

From the factor trees above it is possible to write:

$$\begin{aligned} 36 &= 2 \times 2 \times 3 \times 3 \\ &= 2^2 \times 3^2 \end{aligned}$$

When a number is written in this way, it is said to be written as the *product of its prime factors.*

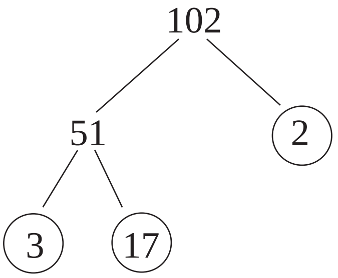
Example 2

Express each of the following numbers as the product of its prime factors:

(a) 102 (b) 60

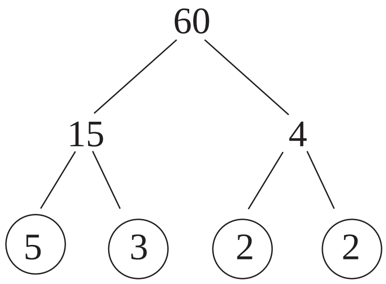
Solution

(a) Start by creating a factor tree:



$102 = 2 \times 3 \times 17$

(b) Start by creating a factor tree:



$60 = 5 \times 3 \times 2 \times 2$

Put the prime numbers in ascending order:

$$\begin{aligned} 60 &= 2 \times 2 \times 3 \times 5 \\ &= 2^2 \times 3 \times 5 \end{aligned}$$

Example 3

A number is expressed as the product of its prime factors as

$$2^3 \times 3^2 \times 5$$

What is the number?

Solution

$$\begin{aligned} 2^3 \times 3^2 \times 5 &= 2 \times 2 \times 2 \times 3 \times 3 \times 5 \\ &= 360 \end{aligned}$$

Exercises

1. Draw factor trees for the following numbers:

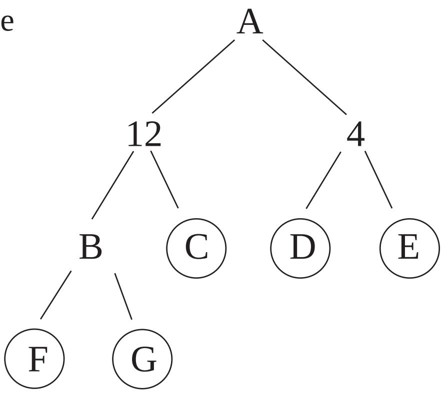
   (a) 20 (b) 100 (c) 88

2. Draw two different factor trees for 40.

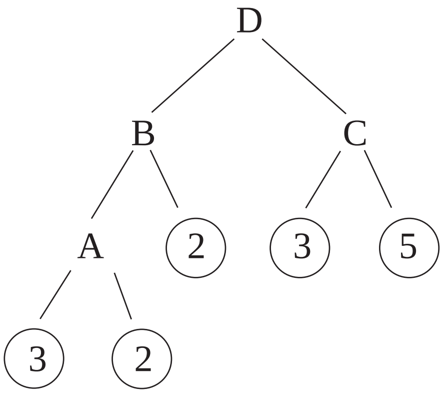
3. (a) Draw two different factor trees for 66.

   (b) Can you draw any other different factor trees for 66?

4. Copy the factor tree opposite and fill in the missing numbers:



5. Fill in the missing numbers on a copy of the factor tree opposite:



6. Use a factor tree to find the prime factors of:

   (a) 30 (b) 80 (c) 200

7. Write each of the following numbers as the product of their prime factors:

   (a) 62 (b) 64 (c) 82

   (d) 320 (e) 90 (f) 120

   (g) 54 (h) 38 (i) 1000

8. A number is expressed as the product of its prime factors as:

$$2^3 \times 3 \times 5^2$$

What is the number?

9. The prime factors of a number are 2, 7 and 11.
   Which are the three *smallest* numbers with these prime factors?

10. Which is the *smallest* number that has:
    (a) 4 different prime factors,
    (b) 5 prime factors?

11. (a) Write down two numbers, neither of which must end in 0, and which multiply together to give 1000.
    (b) Repeat question 11 (a), this time writing down two numbers which multiply to give 1 000 000.

2.3 Index Notation

You will have seen the occasional use of index notation in the last section; for example, in the statement

$$2^3 \times 3^2 \times 5 = 360$$

which contains 2 indices.

We read 2^3 as *"two to the power of three"* or *"two cubed"*: 2 is the *base number,* 3 is the *index.*

In general, a^n is the result of multiplying the base number, a, by itself n times, n being the index.

$$a^n = \underset{\longleftarrow\; n \text{ times} \;\longrightarrow}{a \times a \times a \times \ldots\ldots\ldots \times a \times a \times a \times a}$$

A calculator can be used to work out powers. The index button is usually marked x^y or y^x. Sometimes you will need to press the SHIFT or 2nd FUNCTION key before using the index button. You should find out which buttons you need to use on your calculator.

For example, to calculate 5^4 you may need to press

either $\boxed{5}\ \boxed{x^y}\ \boxed{4}\ \boxed{=}$

or $\boxed{5}\ \boxed{\text{SHIFT}}\ \overset{x^y}{\boxed{}}\ \boxed{4}\ \boxed{=}$

to get the correct answer of 625.

Example 1

Calculate:

(a) 2^4 (b) 7^3 (c) 10^5

Check your answers using a calculator.

Solution

(a) $2^4 = 2 \times 2 \times 2 \times 2$

$= 16$

Using a calculator,

either $\boxed{2}\ \boxed{\text{SHIFT}}\ \overset{x^y}{\boxed{}}\ \boxed{4}\ \boxed{=}$ 16

or $\boxed{2}\ \boxed{x^y}\ \boxed{4}\ \boxed{=}$ 16

(b) $7^3 = 7 \times 7 \times 7$

$= 343$

Using a calculator,

either $\boxed{7}\ \boxed{\text{SHIFT}}\ \overset{x^y}{\boxed{}}\ \boxed{3}\ \boxed{=}$ 343

or $\boxed{7}\ \boxed{x^y}\ \boxed{3}\ \boxed{=}$ 343

(c) $10^5 = 10 \times 10 \times 10 \times 10 \times 10$

$= 100\,000$

Using a calculator,

either $\boxed{1}\ \boxed{0}\ \boxed{\text{SHIFT}}\ \overset{x^y}{\boxed{}}\ \boxed{5}\ \boxed{=}$ 100 000

or $\boxed{1}\ \boxed{0}\ \boxed{x^y}\ \boxed{5}\ \boxed{=}$ 100 000

Example 2

Write these statements, filling in the missing numbers:

(a) $32 = 2^{\square}$ (b) $1000\,000 = 10^{\square}$

Solution

(a) $32 = 2 \times 2 \times 2 \times 2 \times 2$

$= 2^5$

(b) $1\,000\,000 = 10 \times 10 \times 10 \times 10 \times 10 \times 10$

$= 10^6$

Exercises

1. Copy the following statements and fill in the missing numbers:

(a) $6 \times 6 \times 6 \times 6 \times 6 = 6^{\square}$ (b) $3 \times 3 \times 3 \times 3 = 3^{\square}$

(c) $7 \times 7 \times 7 \times 7 \times 7 \times 7 = 7^{\square}$ (d) $9 \times 9 \times 9 \times 9 \times 9 = 9^{\square}$

2. Calculate:

(a) 2^3 (b) 3^3 (c) 10^4

(d) 5^3 (e) 2^7 (f) 3^4

(g) 9^2 (h) 10^3 (i) 10^7

3. Copy the following statements and fill in the missing numbers:

(a) $100 = 10^{\square}$ (b) $81 = \square^{2}$ (c) $81 = \square^{4}$

(d) $16 = \square^{4}$ (e) $16 = \square^{2}$ (f) $7^{\square} = 2401$

4. Calculate:

(a) $5^2 \times 2^2$ (b) $3^2 \times 2^4$ (c) $7^2 \times 2^3$

(d) $6^2 \times 2$ (e) $9^2 \times 3$ (f) $5^3 \times 2^3$

5. Copy each of the following statements and fill in the missing numbers:

(a)
$$\begin{aligned} 2^3 \times 2^5 &= (2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2 \times 2) \\ &= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \\ &= 2^{\square} \end{aligned}$$

(b)
$$\begin{aligned} 5^7 \times 5^2 &= (5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5) \times (5 \times 5) \\ &= 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \\ &= 5^{\square} \end{aligned}$$

(c) $6^4 \times 6^2 = 6^{\square}$

(d) $7^3 \times 7^7 = 7^{\square}$

(e) $8^6 \times 8 = 8^{\square}$

6. Copy the following statements and fill in the missing numbers:

(a)
$$\begin{aligned} 9^3 &= 9 \times 9 \times 9 \\ &= (3 \times 3) \times (3 \times 3) \times (3 \times 3) \\ &= 3 \times 3 \times 3 \times 3 \times 3 \times 3 \\ &= 3^{\square} \end{aligned}$$

(b) $9^4 = 3^{\square}$

7. If $4^2 = 2^n$, which number does n represent? Answer this question using a similar method to the one used in question 6.

8. If $4^n = 2^{12}$, which number does n represent?

9. If $125^4 = 5^n$, which number does n represent?

10. Copy the following statements and fill in the missing numbers:

(a) $5^4 \times 2^4 = 10^{\square}$

(b) $3^5 \times 2^5 = 6^{\square}$

(c) $4^{\square} \times 2^3 = 8^3$

(d) $7^5 \times 3^5 = 21^{\square}$

(e) $7^4 \times \square^4 = 28^4$

(f) $5^9 \times \square^9 = 10^9$

2.4 Highest Common Factor and Lowest Common Multiple

The *highest common factor* (HCF) of two numbers is the largest number that is a factor of both.

The factors of 12 are 1, 2, ③, 4, 6, 12.

The factors of 15 are 1, ③, 5, 15.

So the HCF of 12 and 15 is 3.

The HCF is easy to find for some numbers, but for others it is more difficult. In harder cases, the best way to find the HCF is to use prime factors.

Example 1

Find the HCF of:

(a) 20 and 30 (b) 14 and 12

Solution

(a) The factors of 20 are 1, 2, 4, 5, 10 and 20.

The factors of 30 are 1, 2, 3, 5, 6, 10, 15 and 30.

The HCF of 20 and 30 is 10.

(b) The factors of 14 are 1, 2, 7 and 14.

The factors of 12 are 1, 2, 3, 4, 6 and 12.

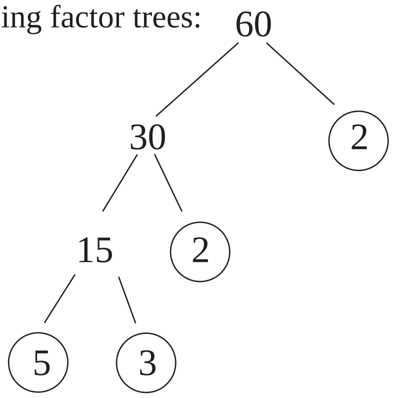
The HCF of 14 and 12 is 2.

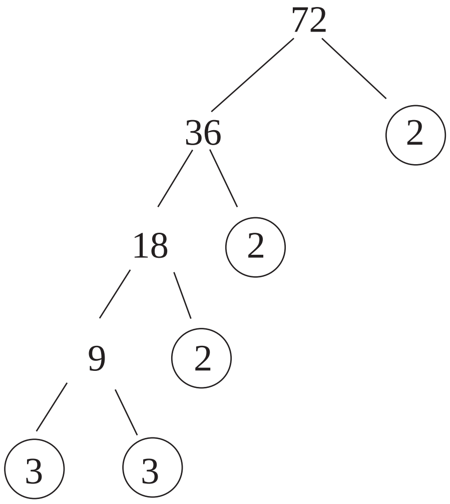
Example 2

Find the HCF of 60 and 72.

Solution

Using factor trees:





$$60 = 2 \times 2 \times 3 \times 5$$
$$= 2^2 \times 3 \times 5$$

$$72 = 2 \times 2 \times 2 \times 3 \times 3$$
$$= 2^3 \times 3^2$$

The HCF is calculated using the prime factors that are common to both numbers. In this case, 2 appears twice in both, and 3 appears once in both.

So,

$$\text{the HCF of 60 and 72} = 2 \times 2 \times 3$$
$$= 12$$

To be in the HCF, the prime factor must be in *both* lists:

$$\begin{array}{rcllllll} 60 &=& 2 &\times 2 & & \times 3 & & \times 5 \\ 72 &=& 2 &\times 2 &\times 2 &\times 3 &\times 3 & \\ \hline \text{HCF} &=& 2 &\times 2 & &\times 3 & & \end{array}$$

$$\text{HCF} = 12$$

Alternatively, using indices:

$$60 \;=\; 2^2 \times 3^1 \times 5^1$$

$$72 \;=\; 2^3 \times 3^2 \times 5^0$$

Lowest power of 3

$$\text{HCF} \;=\; 2^2 \times 3^1 \times 5^0$$

Lowest power of 2 *Lowest power of 5*

$$\text{HCF} \;=\; 12$$

The *lowest common multiple* (LCM) of two numbers is the smallest number that is a multiple of both.

For example, 18 is the smallest number that is a multiple of both 6 and 9, so the LCM of 6 and 9 is 18.

Example 3

What is the LCM of:

(a) 5 and 7 (b) 6 and 10

Solution

(a) The multiples of 5 are:

$5,\ 10,\ 15,\ 20,\ 25,\ 30,\ \textcircled{35,}\ 40,\ 45,\ \ldots$

The multiples of 7 are:

$7,\ 14,\ 21,\ 28,\ \textcircled{35,}\ 42,\ 49,\ \ldots$

The LCM of 5 and 7 = 35.

(b) The multiples of 6 are:

$6,\ 12,\ 18,\ 24,\ \textcircled{30,}\ 36,\ 42,\ \ldots$

The multiples of 10 are:

$10,\ 20,\ \textcircled{30,}\ 40,\ 50,\ 60,\ \ldots$

The LCM of 6 and 10 = 30.

The LCM for larger numbers can be found by using prime factorisation.

Example 4

Find the LCM of 60 and 72.

Solution

From Example 2,

$60 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3 \times 5$ and $72 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2$

The LCM includes all the factors from either number.

To be in the LCM, the prime factor can be in *either* list or in *both* lists:

$$\begin{array}{rcl} 60 & = & 2 \times 2 \qquad \times 3 \qquad \times 5 \\ 72 & = & 2 \times 2 \times 2 \times 3 \times 3 \\ \hline \text{LCM} & = & 2 \times 2 \times 2 \times 3 \times 3 \times 5 \end{array}$$

$\text{LCM} = 360$

Alternatively, using indices:

$60 = 2^2 \times 3^1 \times 5^1$

$72 = 2^3 \times 3^2 \times 5^0$

Highest power of 3

$\text{LCM} = 2^3 \times 3^2 \times 5^1$

Highest power of 2 *Highest power of* 5

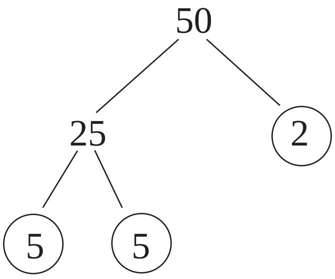
$\text{LCM} = 360$

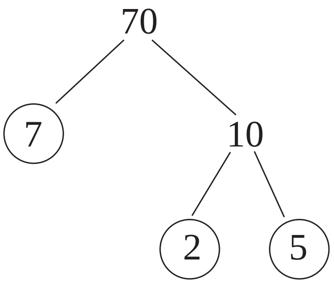
Example 5

Find the HCF and LCM of 50 and 70.

Solution

Using factor trees to find the prime factorisations:





$50 = 2 \times 5 \times 5$

$= 2^1 \times 5^2$

$70 = 2 \times 5 \times 7$

$= 2^1 \times 5^1 \times 7^1$

$$\begin{aligned} \text{HCF} &= 2^1 \times 5^1 \times 7^0 \\ &= 10 \end{aligned}$$

$$\begin{array}{rcl} \text{HCF} & = & 2 \times 5 \\ \hline 50 & = & 2 \times 5 \times 5 \\ 70 & = & 2 \times 5 \qquad \times 7 \\ \hline \text{LCM} & = & 2 \times 5 \times 5 \times 7 \end{array}$$

$$\begin{aligned} \text{LCM} &= 2^1 \times 5^2 \times 7^1 \\ &= 350 \end{aligned}$$

Exercises

1. (a) List the factors of 21.
   (b) List the factors of 35.
   (c) What is the HCF of 21 and 35 ?

2. Find the HCF of:
   (a) 6 and 9
   (b) 14 and 18
   (c) 30 and 24
   (d) 15 and 10

3. (a) Use a factor tree to find the prime factorisation of 42.
   (b) Use a factor tree to find the prime factorisation of 90.
   (c) Find the HCF of 42 and 90.

4. What is the HCF of:
   (a) 90 and 120
   (b) 96 and 72
   (c) 56 and 60
   (d) 77 and 50
   (e) 300 and 550
   (f) 320 and 128 ?

5. (a) List the first 10 multiples of 8.
   (b) List the first 10 multiples of 6.
   (c) What is the LCM of 6 and 8 ?

6. What is the LCM of:
   (a) 5 and 3
   (b) 9 and 6
   (c) 8 and 10
   (d) 12 and 9
   (e) 15 and 20
   (f) 6 and 11 ?

7. (a) Use a factor tree to find the prime factorisation of 66.
   (b) Use a factor tree to find the prime factorisation of 40.
   (c) Find the LCM of 40 and 66.

8. Find the LCM of:

    (a) 28 and 30 (b) 16 and 24 (c) 20 and 25
    (d) 60 and 50 (e) 12 and 18 (f) 21 and 35

9. Two lighthouses can be seen from the top of a hill. The first flashes once every 8 seconds, and the other flashes once every 15 seconds. If they flash simultaneously, how long is it until they flash again at the same time?

10. At a go-kart race track, Vic completes a lap in 40 seconds; Paul completes a lap in 30 seconds, and Mark completes a lap in 50 seconds.

    If all three start a lap at the same time, how long is it before

    (a) Paul overtakes Vic,
    (b) Vic overtakes Mark?

2.5 Squares and Square Roots

To *square* a number you multiply the number by itself.

If you square 8, you multiply 8 by 8:

$$8 \times 8 = 64$$

so the square of 8 = 64.

The calculator button for squaring numbers usually looks like $\boxed{x^2}$ or $\overset{x^2}{\boxed{}}$. For the second type of calculator you have to press the SHIFT or 2nd FUNCTION key first.

Sometimes we need to answer questions such as,

"What number was squared to get 64?"

When answering this we need to use square roots.

The *square root* of a number is a number which, when squared (multiplied by itself), gives you the first number.

The sign $\sqrt{}$ means *square root.*

We say that:

the square root of 64 is 8, i.e. $\sqrt{64} = 8$

since the square of 8 is 64, i.e. $8^2 = 64$

The calculator button for finding a square root usually looks like $\boxed{\sqrt{\ }}$.
With some calculators you press the square root button *before* entering the number; with others you enter the number and *then* press the square root button. You need to find out how your calculator works out square roots.

Example 1

(a) Square each of these numbers:

1, 5, 7, 14

(b) Find:

$\sqrt{25}$, $\sqrt{49}$, $\sqrt{196}$, $\sqrt{1}$

Solution

(a)
$$1^2 = 1 \times 1 = 1$$
$$5^2 = 5 \times 5 = 25$$
$$7^2 = 7 \times 7 = 49$$
$$14^2 = 14 \times 14 = 196$$

(b)
$\sqrt{25} = 5$ because $5^2 = 25$

$\sqrt{49} = 7$ because $7^2 = 49$

$\sqrt{196} = 14$ because $14^2 = 196$

$\sqrt{1} = 1$ because $1^2 = 1$

Example 2

Use your calculator to find:

(a) 54^2

(b) $\sqrt{961}$

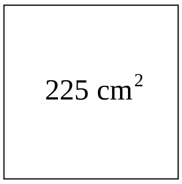
Solution

(a) Either $\boxed{5}\ \boxed{4}\ \boxed{x^2}\ \boxed{=}$ 2916 *or* $\boxed{5}\ \boxed{4}\ \boxed{\text{SHIFT}}\ \overset{x^2}{\boxed{}}\ \boxed{=}$ 2916

(b) Either $\boxed{9}\ \boxed{6}\ \boxed{1}\ \boxed{\sqrt{\ }}\ \boxed{=}$ 31 *or* $\boxed{\sqrt{\ }}\ \boxed{9}\ \boxed{6}\ \boxed{1}\ \boxed{=}$ 31

Example 3

The area of this square is $225\ \text{cm}^2$.
What is the length of each side?



Solution

$$\text{Area} = (\text{length of side})^2$$

$$225\ \text{cm}^2 = (\text{length of side})^2$$

$$\text{Length of side} = \sqrt{225}$$

$$= 15\ \text{cm}$$

Example 4

Use your calculator to find $\sqrt{5}$ correct to 2 decimal places.

Solution

$$\sqrt{5} = 2.236067977$$

$$= 2.24 \quad \text{correct to 2 decimal places.}$$

Exercises

1. (a) Square these numbers:

   2, 4, 9, 11, 12, 18, 20

   (b) Use your answers to (a) to find:

   $\sqrt{144}$, $\sqrt{16}$, $\sqrt{121}$, $\sqrt{4}$, $\sqrt{81}$, $\sqrt{400}$, $\sqrt{324}$

2. Write down the following square roots *without* using a calculator:

   (a) $\sqrt{9}$ (b) $\sqrt{36}$ (c) $\sqrt{100}$

   (d) $\sqrt{169}$ (e) $\sqrt{225}$ (f) $\sqrt{0}$

3. Use a calculator to find these square roots, giving your answers correct to 2 decimal places:

   (a) $\sqrt{6}$ (b) $\sqrt{10}$ (c) $\sqrt{12}$

   (d) $\sqrt{20}$ (e) $\sqrt{50}$ (f) $\sqrt{90}$

4. What are the lengths of the sides of a square which has an area of 81 cm^2 ?

5. A square has an area of 140 cm^2. How long are the sides of this square, to the nearest mm?

6. Explain why $7 < \sqrt{51} < 8$.

7. Copy the statements below and complete each one, putting two *consecutive* whole numbers in the empty spaces:

   (a) $\square < \sqrt{70} < \square$ (b) $\square < \sqrt{90} < \square$

   (c) $\square < \sqrt{5} < \square$ (d) $\square < \sqrt{2} < \square$

   (e) $\square < \sqrt{115} < \square$ (f) $\square < \sqrt{39} < \square$

8. Decide whether each of these statements is *true* or *false*:

   (a) $4 < \sqrt{10} < 5$ (b) $2.6 < \sqrt{7} < 2.7$

   (c) $3.4 < \sqrt{12} < 3.5$ (d) $3.7 < \sqrt{15} < 3.8$

   Write correct statements to replace those that are *false*, but keep the same square roots in them.

9. What is the perimeter of a square with area 196 cm^2 ?

10. Three identical squares are put side-by-side to form a rectangle. The area of the rectangle is 192 cm^2. What are the lengths of the sides of the rectangle?